

INDEX MAPS IN THE K -THEORY OF GRAPH ALGEBRAS

TOKE MEIER CARLSEN, SØREN EILERS, AND MARK TOMFORDE

ABSTRACT. Let $C^*(E)$ be the graph C^* -algebra associated to a graph E and let J be a gauge-invariant ideal in $C^*(E)$. We compute the cyclic six-term exact sequence in K -theory associated to the extension

$$0 \longrightarrow J \longrightarrow C^*(E) \longrightarrow C^*(E)/J \longrightarrow 0$$

in terms of the adjacency matrix associated to E . The ordered six-term exact sequence is a complete stable isomorphism invariant for several classes of graph C^* -algebras, for instance those containing a unique proper nontrivial ideal. Further, in many other cases, finite collections of such sequences comprise complete invariants.

Our results allow for explicit computation of the invariant, giving an exact sequence in terms of kernels and cokernels of matrices determined by the vertex matrix of E .

1. INTRODUCTION

The cyclic six-term exact sequence

$$(1.1) \quad \begin{array}{ccccc} K_0(J) & \xrightarrow{\iota_*} & K_0(C^*(E)) & \xrightarrow{\pi_*} & K_0(C^*(E)/J) \\ \partial_1 \uparrow & & & & \downarrow \partial_0 \\ K_1(C^*(E)/J) & \xleftarrow{\pi_*} & K_1(C^*(E)) & \xleftarrow{\iota_*} & K_1(J) \end{array}$$

is a complete stable isomorphism invariant for a graph C^* -algebra $C^*(E)$ of real rank zero containing a proper nontrivial ideal J when any of the following are satisfied

- J is the unique proper nontrivial ideal of $C^*(E)$ ([7, Theorem 4.5]),
- J is a smallest proper nontrivial ideal of $C^*(E)$, and $C^*(E)/J$ is AF ([6, Corollary 6.4]),
- J is a largest proper nontrivial ideal of $C^*(E)$, and J is AF ([7, Theorem 4.7]).

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In other cases (cf. [6]) a complete invariant may be obtained by combining several six-term exact sequences associated to $C^*(E)$ and its ideals.

It is therefore important to address how to compute sequences of the form in (1.1). In the existing literature it is shown that if E is a row-finite graph with no sinks, then

$$K_0(C^*(E)) \cong \operatorname{coker}(A^t - I) \text{ and } K_1(C^*(E)) \cong \ker(A^t - I),$$

where $A^t - I : \mathbb{Z}^{E^0} \rightarrow \mathbb{Z}^{E^0}$ is the linear map given by the transpose of the vertex matrix A of E minus the identity matrix I . This description of the K_0 -group also includes a description of its order, and a similar computation exists when sinks and infinite emitters are allowed. Since gauge-invariant ideals of graph C^* -algebras and the corresponding quotients are naturally isomorphic to graph C^* -algebras, this allows one to compute the K_0 -groups and K_1 -groups in the above exact sequence. Moreover, since the C^* -algebra of a graph satisfying Condition (K) has real rank zero [9, Theorem 3.5], it follows from [3] that the descending connecting map $\partial_0 : K_0(C^*(E)/J) \rightarrow K_1(J)$ is the zero map. All that remains is to describe a method for computing the other connecting group homomorphisms.

The purpose of this paper is to provide explicit formulae for computing the six-term exact sequence, the main challenge being to compute the connecting map $\partial_1 : K_1(C^*(E)/J) \rightarrow K_0(J)$. We shall also show that $\partial_0 : K_0(C^*(E)/J) \rightarrow K_1(J)$ is the zero map regardless of whether the graph E satisfies Condition (K) or not. All our calculations hold for an arbitrary graph algebra $C^*(E)$ and an arbitrary gauge-invariant ideal J in $C^*(E)$, even in the case of so-called *breaking vertices*.

To compute ∂_1 , we need to choose generators for the K -groups involved. There is a canonical (and well-known) way to do this in K_0 ; one can choose an isomorphism of $K_0(C^*(E))$ with $\operatorname{coker}(A^t - I)$ taking $[p_v]$ to $\mathbf{e}_v + \operatorname{Im}(A^t - I)$, where \mathbf{e}_v is the vector with a 1 in the v^{th} position and zeroes elsewhere. However, for the K_1 -group the calculation is substantially harder. Descriptions of K_1 can be found in [2] and [5], but we need a more explicit description and therefore choose a different approach, choosing explicit generators for K_1 based on a slightly intricate indexing of the entries in a matrix over $C^*(E)$. Although any quotient of a graph C^* -algebra by a gauge-invariant ideal is isomorphic to a graph C^* -algebra, it will be more convenient for us to use that such a quotient is isomorphic to a relative graph C^* -algebra (cf. [11]), and we will therefore find generators of K_0 and K_1 , not just for graph C^* -algebras, but for relative graph C^* -algebras.

We prove that the generators we choose for K_1 are indeed generators by computing the index map of the canonical Toeplitz extension of $C^*(E)$, using methods developed by Katsura in that framework. Our approach involves computing the index map using the canonical method (cf. [14]) of lifting the generating unitaries to partial isometries and computing defects. This method has similarities with the approach for Cuntz-Krieger algebras outlined by Cuntz himself in [4], and discussed with a few more details

in [13]. After describing how to choose generators for K_0 and K_1 of any relative graph C^* -algebra, we determine the index map $\partial_1 : K_1(C^*(E)/J) \rightarrow K_0(J)$ by, in a new extension, again lifting our generating unitaries to partial isometries, and computing defects.

In Section 2 we briefly introduce graph C^* -algebras, relative graph C^* -algebras, and gauge-invariant ideals of graph C^* -algebras. In Section 3 we find generators of K_0 and K_1 of any relative graph C^* -algebra. Section 4 states the main result of the paper, allowing the computation of the index map $\partial_1 : K_1(C^*(E)/J) \rightarrow K_0(J)$ and the other maps in the six-term exact sequence (1.1), and this result is proved in Section 5.

2. PRELIMINARIES

A (directed) graph $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s : E^1 \rightarrow E^0$ identifying the range and source of each edge. A vertex $v \in E^0$ is called a *sink* if $|s^{-1}(v)| = 0$, and v is called an *infinite emitter* if $|s^{-1}(v)| = \infty$. A graph E is said to be *row-finite* if it has no infinite emitters. If v is either a sink or an infinite emitter, then we call v a *singular vertex*. We write E_{sing}^0 for the set of singular vertices. Vertices that are not singular vertices are called *regular vertices* and we write E_{reg}^0 for the set of regular vertices.

If E is a graph, a *Cuntz-Krieger E -family* is a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges which satisfy the *Cuntz-Krieger relations*:

- (CK1) $s_e^* s_e = p_{r(e)}$ for every $e \in E^1$;
- (CK2) $p_v = \sum_{s(e)=v} s_e s_e^*$ for every $v \in E_{\text{reg}}^0$;
- (CK3) $s_e s_e^* \leq p_{s(e)}$ for every $e \in E^1$.

The *graph algebra* $C^*(E)$ is defined to be the C^* -algebra generated by a universal Cuntz-Krieger E -family.

It will in this paper also be relevant to work with *relative graph C^* -algebras* introduced in [11]. To define a relative graph C^* -algebra we must, in addition to a graph E , specify a subset R of E_{reg}^0 . A *Cuntz-Krieger (E, R) -family* is then a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges which satisfy the relations (CK1) and (CK3) above together with the following relative Cuntz-Krieger relation:

$$(RCK2) \quad p_v = \sum_{s(e)=v} s_e s_e^* \text{ for every } v \in R.$$

The *relative graph algebra* $C^*(E, R)$ is defined to be the C^* -algebra generated by a universal Cuntz-Krieger (E, R) -family. If $R = E_{\text{reg}}^0$, then a Cuntz-Krieger (E, R) -family is the same as a Cuntz-Krieger E -family and $C^*(E, R) = C^*(E)$. If $R = \emptyset$, then $C^*(E, R)$ is the Toeplitz algebra $\mathcal{T}(E)$ defined in [8, Theorem 4.1]. We will call a Cuntz-Krieger (E, \emptyset) -family a *Toeplitz-Cuntz-Krieger E -family*.

A *path* in E is a sequence of edges $\alpha = \alpha_1 \alpha_2 \dots \alpha_n$ with $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < n$, and we say that α has length $|\alpha| = n$. We let E^n denote the set of all paths of length n , and we let $E^* := \bigcup_{n=0}^{\infty} E^n$ denote the set of finite paths in E . Note that vertices are considered paths of length zero. The maps r, s extend to E^* , and for $v, w \in E^0$ we write $v \geq w$ if there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. Also for a path $\alpha := \alpha_1 \dots \alpha_n$ we define $s_\alpha := s_{\alpha_1} \dots s_{\alpha_n}$, and for a vertex $v \in E^0$ we let $s_v := p_v$. It is a consequence of the relations (CK1) and (CK3) that $C^*(E, R) = \overline{\text{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}$.

We say that a path $\alpha := \alpha_1 \dots \alpha_n$ of length 1 or greater is a *cycle* if $r(\alpha) = s(\alpha)$, and we call the vertex $s(\alpha) = r(\alpha)$ the *base point* of the cycle. A cycle is said to be *simple* if $s(\alpha_i) \neq s(\alpha_1)$ for all $1 < i \leq n$. The following is an important condition in the theory of graph C^* -algebras.

Condition (K): No vertex in E is the base point of exactly one simple cycle; that is, every vertex is either the base point of no cycles or at least two simple cycles.

For any graph E a subset $H \subseteq E^0$ is *hereditary* if whenever $v, w \in E^0$ with $v \in H$ and $v \geq w$, then $w \in H$. A hereditary subset H is *saturated* if whenever $v \in E_{\text{reg}}^0$ with $r(s^{-1}(v)) \subseteq H$, then $v \in H$. For any saturated hereditary subset H , the *breaking vertices* corresponding to H are the elements of the set

$$B_H := \{v \in E^0 : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}.$$

An *admissible pair* (H, S) consists of a saturated hereditary subset H and a subset $S \subseteq B_H$. For a fixed graph E we order the collection of admissible pairs for E by defining $(H, S) \leq (H', S')$ if and only if $H \subseteq H'$ and $S \subseteq H' \cup S'$. For any admissible pair (H, S) we define $J_{(H, S)}$ to be the ideal in $C^*(E)$ generated by

$$\{p_v : v \in H\} \cup \{p_{v_0}^H : v_0 \in S\},$$

where $p_{v_0}^H$ is the *gap projection* defined by

$$p_{v_0}^H := p_{v_0} - \sum_{\substack{s(e)=v_0 \\ r(e) \notin H}} s_e s_e^*.$$

Note that the definition of B_H ensures that the sum on the right is finite.

For any graph E there is a canonical gauge action $\gamma : \mathbb{T} \rightarrow \text{Aut } C^*(E)$ with the property that for any $z \in \mathbb{T}$ we have $\gamma_z(p_v) = p_v$ for all $v \in E^0$ and $\gamma_z(s_e) = z s_e$ for all $e \in E^1$. We say that an ideal $J \triangleleft C^*(E)$ is *gauge invariant* if $\gamma_z(J) \subseteq J$ for all $z \in \mathbb{T}$.

There is a bijective correspondence between the lattice of admissible pairs of E and the lattice of gauge-invariant ideals of $C^*(E)$ given by $(H, S) \mapsto J_{(H, S)}$ [2, Theorem 3.6]. When E satisfies Condition (K), all ideals of $C^*(E)$

are gauge invariant [2, Corollary 3.8] and the map $(H, S) \mapsto J_{(H, S)}$ is onto the lattice of ideals of $C^*(E)$. When $B_H = \emptyset$, we write J_H in place of $J_{(H, \emptyset)}$ and observe that J_H equals the ideal generated by $\{p_v : v \in H\}$. Note that if E is row-finite, then B_H is empty for every saturated hereditary subset H .

3. K -THEORY FOR RELATIVE GRAPH ALGEBRAS

For a graph E , the *adjacency matrix* is the $E^0 \times E^0$ matrix A_E with

$$A_E(v, w) := \# \{e \in E^1 : s(e) = v \text{ and } r(e) = w\}.$$

Note that the entries of A_E are elements of $\{0, 1, 2, \dots\} \cup \{\infty\}$. Writing the adjacency matrix with respect to the decomposition $E^0 = E_{\text{reg}}^0 \sqcup E_{\text{sing}}^0$, where the regular vertices are listed first, we obtain a (possibly infinite) block matrix

$$A_E = \begin{bmatrix} A & \alpha \\ H & \eta \end{bmatrix}$$

in which all entries of A and α are finite, but the entries in H and η may be infinite. We will often just substitute “ $*$ ” for H and η , as they turn out to be irrelevant for the K -theory. Indeed, by [2] and [5] we know that the map

$$\begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} : \mathbb{Z}^{E_{\text{reg}}^0} \rightarrow \mathbb{Z}^{E^0}$$

contains the needed information, as

$$K_0(C^*(E)) \simeq \text{coker} \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} \quad K_1(C^*(E)) \simeq \ker \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}.$$

This result can be generalized to relative graph C^* -algebras. In fact, we prove in Proposition 3.8 that if E is a graph, $R \subseteq E_{\text{reg}}$, and $A_E = \begin{bmatrix} A & \alpha \\ H & \eta \end{bmatrix}$ is the adjacency matrix of E written with respect to the decomposition $E^0 = R \sqcup (E^0 \setminus R)$, where the vertices belonging to R are listed first, then there exists a group isomorphism $\chi_0 : \text{coker} \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} \rightarrow K_0(C^*(E, R))$ given for any $v \in R$ by

$$\chi_0 \left(\mathbf{e}_v + \text{im} \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} \right) = [p_v]_0,$$

and we construct a similar group isomorphism χ_1 between $\ker \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}$ and $K_1(C^*(E, R))$. For this we first introduce some notation:

Given $\mathbf{x} \in \ker \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}$, first note that by definition \mathbf{x} has only finitely many nonzero entries x_{v_1}, \dots, x_{v_k} . We define

$$\begin{aligned} L_{\mathbf{x}}^+ &:= \{(e, i) : e \in E^1, 1 \leq i \leq -x_{s(e)}\} \cup \{(v, i) : v \in E^0, 1 \leq i \leq x_v\} \\ L_{\mathbf{x}}^- &:= \{(e, i) : e \in E^1, 1 \leq i \leq x_{s(e)}\} \cup \{(v, i) : v \in E^0, 1 \leq i \leq -x_v\} \end{aligned}$$

and note, using the convention that $r(v) = v$ for any $v \in E^0$, that

Lemma 3.1. *When $\mathbf{x} \in \ker \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}$, then for any vertex $v \in E^0$ the sets*

$$L_v^+ = \{(x, i) \in L_{\mathbf{x}}^+ : r(x) = v\}$$

and

$$L_v^- = \{(x, i) \in L_{\mathbf{x}}^- : r(x) = v\}$$

are finite and have the same number of elements.

Proof. We need to consider three cases separately.

CASE I: $v \in R$ and $x_v \geq 0$.

The number of elements in L_v^+ is

$$x_v + \sum_{x_w < 0} \#\{e \in E^1 : s(e) = w, r(e) = v\} \cdot (-x_w) = x_v - \sum_{x_w < 0} A_{v,w}^t x_w$$

and the number of elements in L_v^- is

$$\sum_{x_w > 0} \#\{e \in E^1 : s(e) = w, r(e) = v\} \cdot x_w = \sum_{x_w > 0} A_{v,w}^t x_w$$

so the claim follows by inspecting the v coordinate of the equality $A^t \mathbf{x} = \mathbf{x}$.

CASE II: $v \in R$ and $x_v < 0$.

As above.

CASE III: $v \in E^0 \setminus R$.

The number of elements in L_v^+ is

$$\sum_{x_w < 0} \#\{e \in E^1 : s(e) = w, r(e) = v\} \cdot (-x_w) = - \sum_{x_w < 0} \alpha_{v,w}^t x_w$$

and the number of elements in L_v^- is

$$\sum_{x_w > 0} \#\{e \in E^1 : s(e) = w, r(e) = v\} \cdot x_w = \sum_{x_w > 0} \alpha_{v,w}^t x_w$$

so the claim follows by inspecting the v coordinate of the equality $\alpha^t \mathbf{x} = 0$. \square

Lemma 3.2. *$L_{\mathbf{x}}^+$ and $L_{\mathbf{x}}^-$ are finite sets, and have the same number of elements.*

Proof. This follows from Lemma 3.1, as indeed $L_v^+ \neq \emptyset$ only when v lies in the set

$$\{v : x_v \neq 0\} \cup \{v : x_w \neq 0 \text{ for some } w \in s(r^{-1}(v))\}$$

which is finite since no w is an infinite emitter. \square

Denote the common number of elements in $L_{\mathbf{x}}^+$ and $L_{\mathbf{x}}^-$ by h . Because of Lemma 3.1, we can define bijections

$$[\cdot] : L_{\mathbf{x}}^+ \rightarrow \{1, \dots, h\} \quad \langle \cdot \rangle : L_{\mathbf{x}}^- \rightarrow \{1, \dots, h\}$$

with the property that

$$(3.1) \quad [x, i] = \langle y, j \rangle \implies r(x) = r(y)$$

with the convention $r(v) = v$.

When \mathfrak{A} is a C^* -algebra then we let $M_h(\mathfrak{A})$ denote the C^* -algebra of $h \times h$ -matrices over \mathfrak{A} . We are ready for our key definitions:

Definition 3.3. Suppose that \mathfrak{A} is a C^* -algebra which contains a Toeplitz-Cuntz-Krieger E -family $\{p_v : v \in E^0\} \cup \{s_e : e \in E^1\}$. With notation as above, we define the two elements $V, P \in M_h(\mathfrak{A})$ by

$$V = \sum_{\substack{1 \leq i \leq x_w \\ s(e)=w}} s_e E_{[w,i],\langle e,i \rangle} + \sum_{\substack{1 \leq i \leq -x_w \\ s(e)=w}} s_e^* E_{[e,i],\langle w,i \rangle}$$

and

$$P = \sum_{1 \leq i \leq x_w} p_w E_{[w,i],[w,i]} + \sum_{\substack{1 \leq i \leq -x_w \\ s(e)=w, r(e)=v}} p_v E_{[e,i],[e,i]}.$$

Here $E_{\bullet,\bullet}$ denote the standard matrix units in $M_h(M(\mathfrak{A}))$ where $M(\mathfrak{A})$ is the multiplier algebra of \mathfrak{A} .

Lemma 3.4. If $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a Toeplitz-Cuntz-Krieger E -family, then

$$(3.2) \quad P = \sum_{1 \leq i \leq -x_w} p_w E_{\langle w,i \rangle, \langle w,i \rangle} + \sum_{\substack{1 \leq i \leq x_w \\ s(e)=w, r(e)=v}} p_v E_{\langle e,i \rangle, \langle e,i \rangle},$$

$$(3.3) \quad V^* = \sum_{\substack{1 \leq i \leq x_w \\ s(e)=w}} s_e^* E_{\langle e,i \rangle, [w,i]} + \sum_{\substack{1 \leq i \leq -x_w \\ s(e)=w}} s_e E_{\langle w,i \rangle, [e,i]},$$

$$(3.4) \quad VV^* = \sum_{\substack{1 \leq i \leq x_w \\ s(e)=w}} s_e s_e^* E_{[w,i],[w,i]} + \sum_{\substack{1 \leq i \leq -x_w \\ s(e)=w, r(e)=v}} p_v E_{[e,i],[e,i]},$$

$$(3.5) \quad V^*V = \sum_{\substack{1 \leq i \leq -x_w \\ s(e)=w}} s_e s_e^* E_{\langle w,i \rangle, \langle w,i \rangle} + \sum_{\substack{1 \leq i \leq x_w \\ s(e)=w, r(e)=v}} p_v E_{\langle e,i \rangle, \langle e,i \rangle}.$$

Proof. It follows from Lemma 3.2 and Equation (3.1) that

$$\sum_{(x,i) \in L_{\mathbf{x}}^+} p_{r(x)} E_{[x,i],[x,i]} = \sum_{(x,i) \in L_{\mathbf{x}}^-} p_{r(x)} E_{\langle x,i \rangle, \langle x,i \rangle},$$

and it is easy to check that

$$P = \sum_{(x,i) \in L_{\mathbf{x}}^+} p_{r(x)} E_{[x,i],[x,i]}$$

and that

$$\sum_{(x,i) \in L_{\mathbf{x}}^-} p_{r(x)} E_{\langle x,i \rangle, \langle x,i \rangle} = \sum_{1 \leq i \leq -x_w} p_w E_{\langle w,i \rangle, \langle w,i \rangle} + \sum_{\substack{1 \leq i \leq x_w \\ s(e)=w, r(e)=v}} p_v E_{\langle e,i \rangle, \langle e,i \rangle}$$

from which Equation (3.2) then follows. Equation (3.3) is straightforward to check. For Equation (3.4), using only (3.3) and the matrix unit relations we get that

$$VV^* = \sum_{\substack{1 \leq i \leq x_w \\ s(e)=w}} s_e s_e^* E_{[w,i],[w,i]} + \sum_{\substack{1 \leq i \leq -x_w \\ s(e)=w, r(e)=v \\ s(e')=w, r(e')=v'}} s_e^* s_{e'} E_{[e,i],[e',i]}$$

and (3.4) holds from (CK1) and the fact that the s_e 's have mutually orthogonal ranges. The computation for V^*V is similar. \square

Lemma 3.5. *If $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a Toeplitz-Cuntz-Krieger E -family, then V is a partial isometry with $PV = VP = V$.*

Proof. Using Equation (3.5), the definition of V , and the fact that the s_e 's are partial isometries, we see that $VV^*V = V$, so that V is a partial isometry. Furthermore, (CK3) implies $PV = V$ and $VP = V$ by Equation (3.2). \square

We now let $\{s_e, p_v : e \in E^1, v \in E^0\}$ be the universal Cuntz-Krieger (E, R) -family generating $C^*(E, R)$ and write $V_{\mathbf{x}}$ and $P_{\mathbf{x}}$ for the corresponding elements V and P in $M_h(C^*(E, R))$ defined in Definition 3.3, using the added subscript to emphasize the dependence of each of V and P on $\mathbf{x} \in \ker \begin{bmatrix} A^t - 1 \\ \alpha^t \end{bmatrix}$. In addition, we define $U_{\mathbf{x}} := V_{\mathbf{x}} + (1 - P_{\mathbf{x}})$.

Fact 3.6. *We have that $V_{\mathbf{x}}V_{\mathbf{x}}^* = V_{\mathbf{x}}^*V_{\mathbf{x}} = P_{\mathbf{x}}$, and hence that $U_{\mathbf{x}}$ is a unitary.*

Proof. It follows from Equation (3.4) and (RCK2) that

$$\begin{aligned} V_{\mathbf{x}}V_{\mathbf{x}}^* &= \sum_{1 \leq i \leq x_w} \left(\sum_{s(e)=w} s_e s_e^* \right) E_{[w,i],[w,i]} + \sum_{\substack{1 \leq i \leq -x_w \\ s(e)=w, r(e)=v}} p_v E_{[e,i],[e,i]} \\ &= \sum_{1 \leq i \leq x_w} p_w E_{[w,i],[w,i]} + \sum_{\substack{1 \leq i \leq -x_w \\ s(e)=w, r(e)=v}} p_v E_{[e,i],[e,i]} \\ &= P_{\mathbf{x}} \end{aligned}$$

showing the first claim. Likewise, Equation (3.5) and (RCK2) show that $V_{\mathbf{x}}^*V_{\mathbf{x}} = P_{\mathbf{x}}$. The fact that $U_{\mathbf{x}}$ is a unitary follows. \square

Remark 3.7. Notice that although $U_{\mathbf{x}}$ does depend on the choice of bijections

$$[\cdot] : L_{\mathbf{x}}^+ \rightarrow \{1, \dots, h\} \quad \langle \cdot \rangle : L_{\mathbf{x}}^- \rightarrow \{1, \dots, h\},$$

the element $[U_{\mathbf{x}}]_1$ of $K_1(C^*(E, R))$ does not.

Proposition 3.8. *Let E be a graph, let V be a subset of E_{reg} and let*

$$A_E = \begin{bmatrix} A & \alpha \\ H & \eta \end{bmatrix}$$

be the adjacency matrix of E written with respect to the decomposition $E^0 = V \sqcup (E^0 \setminus V)$ where the vertices belonging to V are listed first.

(1) *There exists a group isomorphism $\chi_0 : \text{coker} \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} \rightarrow K_0(C^*(E, R))$ given for any $v \in E^0$ by*

$$(3.6) \quad \chi_0 \left(\mathbf{e}_v + \text{im} \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} \right) = [p_v]_0.$$

The preimage of the positive cone of $K_0(C^(E, R))$ is generated by*

$$\left\{ \mathbf{e}_v : v \in E^0 \right\} \cup \left\{ \mathbf{e}_v - \sum_{e \in F} \mathbf{e}_{r(e)} : v \in E_{\text{sing}}^0, F \subseteq s^{-1}(v), F \text{ finite} \right\}.$$

(2) *The map $\chi_1 : \ker \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} \rightarrow K_1(C^*(E, R))$ given by*

$$\chi_1(\mathbf{x}) = [\mathbf{U}_{\mathbf{x}}]_1$$

is group isomorphism.

Proof. As noted in [11], we can realize $C^*(E, R)$ as a relative Cuntz-Pimsner algebra over a Hilbert bimodule \mathcal{X}_E . It is not difficult to check that the corresponding Toeplitz algebra $\mathcal{T}_{\mathcal{X}_E}$ is isomorphic to the Toeplitz algebra $\mathcal{T}(E)$. We let $\pi : \mathcal{T}(E) \rightarrow C^*(E, R)$ denote the canonical map, so that

$$(3.7) \quad 0 \longrightarrow \ker \pi \xrightarrow{\iota} \mathcal{T}(E) \xrightarrow{\pi} C^*(E, R) \longrightarrow 0$$

is exact. The associated K -theory is then

$$\begin{array}{ccccc} K_0(\ker \pi) & \xrightarrow{\iota_*} & K_0(\mathcal{T}(E)) & \xrightarrow{\pi_*} & K_0(C^*(E, R)) \\ \partial_1 \uparrow & & & & \downarrow \\ K_1(C^*(E, R)) & \xleftarrow{\pi_*} & K_1(\mathcal{T}(E)) & \xleftarrow{\iota_*} & K_1(\ker \pi). \end{array}$$

Now we appeal to Katsura's work. It follows from the results of [10, §8], that $\ker \pi$ and $\mathcal{T}(E)$ are KK -equivalent to the commutative AF -algebras $c_0(R)$ and $c_0(E^0)$, respectively, and that there are group isomorphisms $\kappa : K_0(\ker \pi) \rightarrow \mathbb{Z}^R$ and $\lambda : K_0(\mathcal{T}(E)) \rightarrow \mathbb{Z}^{E^0}$ such that the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & K_1(C^*(E, R)) & \xrightarrow{\partial_1} & K_0(\ker \pi) & \xrightarrow{\iota_*} & K_0(\mathcal{T}(E)) & \xrightarrow{\pi_*} K_0(C^*(E, R)) \longrightarrow 0 \\ & \downarrow \kappa & & & & \downarrow \lambda & \\ & \mathbb{Z}^R & \xrightarrow{\begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}} & \mathbb{Z}^{E^0} & & & \end{array}$$

commutes with the top row exact. In [10] concrete $*$ -homomorphisms are given inducing κ and λ , but we do not need them here. All we need is the fact that $\lambda(p_v) = \mathbf{e}_v$ and

$$(3.8) \quad \kappa \left(\left[p_w - \sum_{s(e)=w} s_e s_e^* \right]_0 \right) = \mathbf{e}_w$$

for $v \in E^0$ and $w \in R$. It follows that $\pi_* \circ \lambda^{-1}$ is a surjective group homomorphism from \mathbb{Z}^{E^0} to $K_0(C^*(E, R))$ which for any $v \in E^0$ maps \mathbf{e}_v to $[p_v]_0$ and whose kernel is $\text{im} \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}$. The existence of a group isomorphism $\chi_0 : \text{coker} \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} \rightarrow K_0(C^*(E, R))$ which for any $v \in E^0$ satisfies Equation (3.6) follows from this. The description of the positive cone in the row-finite case was given in [1, Theorem 7.1]. For the general situation, it is shown in [15, Theorem 2.2] that the process of desingularization can be used to extend the result from the row-finite case to the general case.

To see that $\chi_1 : \ker \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} \rightarrow K_1(C^*(E, R))$ is a group isomorphism, fix $\mathbf{x} \in \ker \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}$ and lift $\mathbf{U}_{\mathbf{x}} = \mathbf{V}_{\mathbf{x}} + (1 - \mathbf{P}_{\mathbf{x}}) \in M_h(C^*(E, R))$ to $\tilde{\mathbf{U}}_{\mathbf{x}} = \tilde{\mathbf{V}}_{\mathbf{x}} + (1 - \tilde{\mathbf{P}}_{\mathbf{x}}) \in M_h(\mathcal{T}(E))$ where $\tilde{\mathbf{V}}_{\mathbf{x}}$ and $\tilde{\mathbf{P}}_{\mathbf{x}}$ are the elements V and P in $M_h(\mathcal{T}(E))$ we get by using the universal Toeplitz-Cuntz-Krieger E -family which generates $\mathcal{T}(E)$ in Definition 3.3. By Lemma 3.5, $\tilde{\mathbf{V}}_{\mathbf{x}}$ is a partial isometry with $\tilde{\mathbf{P}}_{\mathbf{x}}\tilde{\mathbf{V}}_{\mathbf{x}} = \tilde{\mathbf{V}}_{\mathbf{x}}\tilde{\mathbf{P}}_{\mathbf{x}} = \tilde{\mathbf{V}}_{\mathbf{x}}$. It follows that $\tilde{\mathbf{U}}_{\mathbf{x}}$ is also a partial isometry. We need to compute the defect of $\tilde{\mathbf{U}}_{\mathbf{x}}$ as an element of $K_0(\ker \pi)$. We have by Lemma 3.4 that

$$1 - \tilde{\mathbf{U}}_{\mathbf{x}}\tilde{\mathbf{U}}_{\mathbf{x}}^* = \tilde{\mathbf{P}}_{\mathbf{x}} - \tilde{\mathbf{V}}_{\mathbf{x}}\tilde{\mathbf{V}}_{\mathbf{x}}^* = \sum_{1 \leq i \leq x_w} \left(p_w - \sum_{s(e)=w} s_e s_e^* \right) \mathbf{E}_{[w,i],[w,i]}$$

and a similar equation for $1 - \tilde{\mathbf{U}}_{\mathbf{x}}^*\tilde{\mathbf{U}}_{\mathbf{x}}$. Hence, in $K_0(\ker \pi)$ we have that

$$(3.9) \quad [1 - \tilde{\mathbf{U}}_{\mathbf{x}}\tilde{\mathbf{U}}_{\mathbf{x}}^*]_0 - [1 - \tilde{\mathbf{U}}_{\mathbf{x}}^*\tilde{\mathbf{U}}_{\mathbf{x}}]_0 = \sum_{x_w \neq 0} x_w \left[p_w - \sum_{s(e)=w} s_e s_e^* \right]_0$$

which together with Equation (3.8) and Equation (3.9) implies that

$$(3.10) \quad \kappa \circ \partial_1 \circ \chi_1(\mathbf{x}) = \mathbf{x}$$

for any $\mathbf{x} \in \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}$. This shows that χ_1 is injective. Let us also prove that χ_1 is a group isomorphism. Fix $\mathbf{y} \in K_1(C^*(E, R))$ and note that

$$\begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} \circ \kappa \circ \partial_1(\mathbf{y}) = \lambda \circ \iota_* \circ \partial_1(\mathbf{y}) = 0$$

so that $\mathbf{z} := \kappa \circ \partial_1(\mathbf{y})$ lies in $\ker \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}$. Since $\kappa \circ \partial_1$ is injective, it follows from Equation (3.10) that $\chi_1(\mathbf{z}) = \mathbf{y}$. We conclude that $\kappa \circ \partial_1$ is actually an inverse to χ_1 , and hence χ_1 is a group isomorphism. \square

4. THE INDEX MAP

Let E be a graph and let J be a gauge-invariant ideal in $C^*(E)$. It follows from [2] that J is of the form $J_{(H,S)}$ for an admissible pair (H, S) . Writing

the adjacency matrix of E with respect to the decomposition

$$E_{\text{reg}}^0 \cap H, \quad E_{\text{sing}}^0 \cap H, \quad E_{\text{reg}}^0 \setminus H, \quad E_{\text{sing}}^0 \setminus (H \cup S), \quad S$$

we arrive at the matrix

$$\begin{bmatrix} A & \alpha & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ X & \xi & B & \beta & \eta \\ * & * & * & * & * \\ * & * & \Gamma & \gamma & Z \end{bmatrix}.$$

We are now ready to state our main result. Here and below, whenever $T : G_1 \rightarrow G_2$ is a group homomorphism between abelian groups and H_1 and H_2 are subgroups of G_1 and G_2 , respectively, such that $T(H_1) \subseteq H_2$, then we also use T to denote the group homomorphism from G_1/H_1 to G_2/H_2 induced by T , and we denote by $I_{a_1 \dots a_k}$ the canonical inclusion of the indicated components of a direct sum into a larger direct sum, and by $P_{a_1 \dots a_k}$ the corresponding projection.

Theorem 4.1. *Let E be a graph and let (H, S) be an admissible pair. The six term exact sequence in K -theory*

$$\begin{array}{ccccc} K_0(J_{(H,S)}) & \xrightarrow{\iota_*} & K_0(C^*(E)) & \xrightarrow{\pi_*} & K_0(C^*(E)/J_{(H,S)}) \\ \uparrow \partial_1 & & & & \downarrow \partial_0 \\ K_1(C^*(E)/J_{(H,S)}) & \xleftarrow{\pi_*} & K_1(C^*(E)) & \xleftarrow{\iota_*} & K_1(J_{(H,S)}) \end{array}$$

is isomorphic to

$$\begin{array}{ccccc} \text{coker} \begin{bmatrix} A^t - I \\ \alpha^t \\ 0 \end{bmatrix} & \xrightarrow{\tilde{I}} & \text{coker} \begin{bmatrix} A^t - I & X^t \\ \alpha^t & \xi^t \\ 0 & B^t - I \\ 0 & \beta^t \\ 0 & \eta^t \end{bmatrix} & \xrightarrow{P_{345}} & \text{coker} \begin{bmatrix} B^t - I & \Gamma^t \\ \beta^t & \gamma^t \\ \eta^t & Z^t - I \end{bmatrix} \\ \uparrow \begin{bmatrix} X^t & 0 \\ \xi^t & 0 \\ 0 & I \end{bmatrix} & & & & \downarrow 0 \\ \text{ker} \begin{bmatrix} B^t - I & \Gamma^t \\ \beta^t & \gamma^t \\ \eta^t & Z^t - I \end{bmatrix} & \xleftarrow{I_1 \circ P_2} & \text{ker} \begin{bmatrix} A^t - I & X^t \\ \alpha^t & \xi^t \\ 0 & B^t - I \\ 0 & \beta^t \\ 0 & \eta^t \end{bmatrix} & \xleftarrow{I_1} & \text{ker} \begin{bmatrix} A^t - I \\ \alpha^t \\ 0 \end{bmatrix} \end{array}$$

where \tilde{I} is given by the block matrix

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -\Gamma^t \\ 0 & 0 & -\gamma^t \\ 0 & 0 & I - Z^t \end{bmatrix} = I_{125} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Gamma^t \\ 0 & 0 & \gamma^t \\ 0 & 0 & Z^t \end{bmatrix}.$$

Each cokernel is ordered as described in Theorem 3.8. We postpone the proof of the theorem to the ensuing section, but remark here that the isomorphism between the two six term exact sequences is given by explicit defined maps which are described in the proof.

For now, let us record a number of examples and specializations:

Remark 4.2. If the saturated hereditary subset H has no breaking vertices (this is always the case if E is row-finite), or if $S = \emptyset$, then the six term exact sequence of Theorem 4.1 reduces to

$$(4.1) \quad \begin{array}{ccccc} \text{coker} \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} & \xrightarrow{I_{12}} & \text{coker} \begin{bmatrix} A^t - I & X^t \\ \alpha^t & \xi^t \\ 0 & B^t - I \\ 0 & \beta^t \end{bmatrix} & \xrightarrow{P_{34}} & \text{coker} \begin{bmatrix} B^t - I \\ \beta^t \end{bmatrix} \\ \uparrow \begin{bmatrix} X^t \\ \xi^t \end{bmatrix} & & & & \downarrow 0 \\ \text{ker} \begin{bmatrix} B^t - I \\ \beta^t \end{bmatrix} & \xleftarrow{P_2} & \text{ker} \begin{bmatrix} A^t - I & X^t \\ \alpha^t & \xi^t \\ 0 & B^t - I \\ 0 & \beta^t \end{bmatrix} & \xleftarrow{I_1} & \text{ker} \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}. \end{array}$$

Remark 4.3. Let E be a row-finite graph with no sinks. Then any gauge-invariant ideal in $C^*(E)$ has the form J_H for some saturated hereditary subset H and the six term exact sequence of Theorem 4.1 reduces in this case to

$$\begin{array}{ccccc} \text{coker} [A^t - I] & \xrightarrow{I_1} & \text{coker} \begin{bmatrix} A^t - I & X^t \\ 0 & B^t - I \end{bmatrix} & \xrightarrow{P_2} & \text{coker} [B^t - I] \\ \uparrow [X^t] & & & & \downarrow 0 \\ \text{ker} [B^t - I] & \xleftarrow{P_2} & \text{ker} \begin{bmatrix} A^t - I & X^t \\ 0 & B^t - I \end{bmatrix} & \xleftarrow{I_1} & \text{ker} [A^t - I]. \end{array}$$

Corollary 4.4. *Let E be a graph such that the associated graph C^* -algebra $C^*(E)$ contains a unique proper nontrivial ideal. Then this ideal has the form J_H for some saturated hereditary subset H with no breaking vertices. Consequently, the cyclic six term exact sequence determined by the short exact sequence $0 \rightarrow J_H \rightarrow C^*(E) \rightarrow C^*(E)/J_H \rightarrow 0$ is isomorphic to the cyclic exact sequence described in (4.1).*

Proof. If E has a unique proper nontrivial ideal, then it follows from [7, Lemma 3.1] that the ideal has the form J_H for a saturated hereditary subset H with no breaking vertices. \square

Example 4.5. Consider the class of graphs $E_{x,y,z}$ given by the adjacency matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 1 & 1 & 0 \\ y & 1 & 1 & 1 \\ z & 0 & 1 & 1 \end{bmatrix}$$

where $x, y, z \in \mathbb{N}$. These graphs all satisfy Condition (K) and have one non-trivial saturated hereditary subset (the subset consisting of the first vertex). Thus we are in the situation of Corollary 4.4, with $E_{\text{reg}}^0 = \{v_2, v_3, v_4\}$ and $E_{\text{reg}}^0 = H = \{v_1\}$. Hence the adjacency matrix has the block form

$$\left[\begin{array}{c|c} \alpha & 0 \\ \hline \xi & B \end{array} \right]$$

and the six-term exact sequence is

$$\begin{array}{ccccc} \text{coker } 0_{1 \times 0} & \xrightarrow{I_1} & \text{coker } \begin{bmatrix} x & y & z \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \xrightarrow{P_{234}} & \text{coker } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ \uparrow [x \ y \ z] & & & & \downarrow 0 \\ \text{ker } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \xleftarrow{P_{123}} & \text{ker } \begin{bmatrix} x & y & z \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \xleftarrow{0} & \text{ker } 0_{1 \times 0} \end{array}$$

which simplifies to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{x-z} \mathbb{Z} \longrightarrow \mathbb{Z}/(x-z) \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

when $x \neq z$ and to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

when $z = x$.

The K_0 -group of the ideal is canonically ordered, and the order of the K_0 -group of the quotient is trivial, irrespective of x, y, z . We may hence apply [7] to prove that $C^*(E_{x,y,z}) \otimes \mathbb{K} \simeq C^*(E_{x',y',z'}) \otimes \mathbb{K}$ precisely when $x - z = \pm(x' - z')$.

Example 4.6. Consider the class of graphs $F_{y,z}$ given by the adjacency matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ y & 3 & 1 \\ \infty & z & 3 \end{bmatrix}$$

where $y, z \in \mathbb{N}$. These graphs all satisfy Condition (K) and have one non-trivial saturated hereditary subset $\{v_1\}$ for which $\{v_3\}$ is breaking. We furthermore have that $E_{\text{reg}}^0 = \{v_2\}$ and $E_{\text{sing}}^0 = \{v_1, v_3\}$. If we consider the ideal $J_{(\{v_1\}, \{v_3\})}$, then the adjacency matrix has the block form

$$\left[\begin{array}{c|c|c} * & 0 & 0 \\ \hline \xi & B & \eta \\ \hline * & \Gamma & Z \end{array} \right]$$

which gives

$$\begin{array}{ccccc}
 & & \begin{bmatrix} 1 & 0 \\ 0 & -z \\ 0 & -2 \end{bmatrix} & & \\
 & & \downarrow & & \\
 \text{coker } 0_{2 \times 0} & \xrightarrow{\quad} & \text{coker } \begin{bmatrix} y \\ 2 \\ 1 \end{bmatrix} & \longrightarrow & \text{coker } \begin{bmatrix} 2 & z \\ 1 & 2 \end{bmatrix} \\
 \uparrow \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix} & & & & \downarrow \\
 \text{ker } \begin{bmatrix} 2 & z \\ 1 & 2 \end{bmatrix} & \longleftarrow & \text{ker } \begin{bmatrix} y \\ 2 \\ 1 \end{bmatrix} & \longleftarrow & \text{ker } 0_{2 \times 0}
 \end{array}$$

simplifying to

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} -1 & -2y \\ 0 & 4-z \end{bmatrix}} \mathbb{Z}^2 \longrightarrow \mathbb{Z}_{z-4} \longrightarrow 0$$

when $z \neq 4$ and to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{bmatrix} -2y \\ 1 \end{bmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} -1 & -2y \\ 0 & 0 \end{bmatrix}} \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0$$

when $z = 4$.

In both cases, the K_0 -group of the ideal is ordered by

$$\{(x_1, x_3) : x_3 > 1 \text{ or } [x_3 = 0, x_1 \geq 0]\},$$

having only the trivial automorphism, so the computations combine with [7, Theorem 4.7] to show that $C^*(F_{y,z}) \otimes \mathbb{K} \simeq C^*(F_{y',z'}) \otimes \mathbb{K}$ precisely when $4 - z = \pm(4 - z')$ and $y - y' \in (4 - z)\mathbb{Z}$.

5. PROOF OF MAIN RESULT

The isomorphism of the two six-term exact sequences in Theorem 4.1 is given by the six group isomorphisms $\chi'_0, \chi_0, \chi''_0, \chi'_1, \chi_1, \chi''_1$ defined as follows. If we let $E_{(H,S)}$ be the subgraph of E with vertices $H \cup S$ and edges $s^{-1}(H) \cup (s^{-1}(S) \cap r^{-1}(H))$, then the graph C^* -algebra $C^*(E_{(H,S)})$ is isomorphic to a full corner of $J_{(H,S)}$ via an embedding $\phi : C^*(E_{(H,S)}) \rightarrow J_{(H,S)}$ with $\phi(p_v) = p_v$ for $v \in H$, $\phi(p_{v_0}) = p_{v_0}^H$ for $v_0 \in S$ and $\phi(s_e) = s_e$ for $e \in E_{(H,S)}^1$ (cf. [2]). Notice that $(E_{(H,S)})_{\text{reg}}^0 = E_{\text{reg}}^0 \cap H$ and that $(E_{(H,S)})_{\text{sing}}^0 = (E_{\text{sing}}^0 \cap H) \cup S$. It follows (for example by [12, Proposition 1.2]) that ϕ induces an isomorphism $\phi_* : K_*(C^*(E_{(H,S)})) \rightarrow K_*(J_{(H,S)})$. Thus if we let $\chi_*^{E_{(H,S)}}$ denote the group isomorphisms given by Proposition 3.8 for $C^*(E_{(H,S)})$, then

$$\chi'_0 := \phi_* \circ \chi_0^{E_{(H,S)}} : \text{coker} \begin{bmatrix} A^t - I \\ \alpha^t \\ 0 \end{bmatrix} \rightarrow K_0(J_{(H,S)})$$

and

$$\chi'_1 := \phi_* \circ \chi_1^{E_{(H,S)}} : \text{ker} \begin{bmatrix} A^t - I \\ \alpha^t \\ 0 \end{bmatrix} \rightarrow K_1(J_{(H,S)})$$

are group isomorphisms. Similarly, if we let $E \setminus H$ be the subgraph of E with vertices $E^0 \setminus H$ and edges $r^{-1}(E^0 \setminus H)$, then there is an isomorphism $\psi : C^*(E \setminus H, S) \rightarrow C^*(E)/J_{(H,S)}$ which for any $v \in E^0 \setminus H$ maps p_v to

$p_v + J_{(H,S)}$ and for any $e \in r^{-1}(E^0 \setminus H)$ maps s_e to $s_e + J_{(H,S)}$ (cf. [11, Example 3.10]). Notice that $(E \setminus H)_{\text{reg}}^0 = E_{\text{reg}}^0 \setminus H$ and that $(E \setminus H)_{\text{sing}}^0 = E_{\text{sing}}^0 \setminus H$. Thus if we let $\chi_*^{(E \setminus H, S)}$ denote the group isomorphisms given by Proposition 3.8 for $C^*(E \setminus H, S)$, then

$$\chi_0'' := \psi_* \circ \chi_0^{(E \setminus H, S)} : \text{coker} \begin{bmatrix} B^t - I & \Gamma^t \\ \beta^t & \gamma^t \\ \eta^t & Z^t - I \end{bmatrix} \rightarrow K_0(C^*(E)/J_{(H,S)})$$

and

$$\chi_1'' := \psi_* \circ \chi_1^{(E \setminus H, S)} : \ker \begin{bmatrix} B^t - I & \Gamma^t \\ \beta^t & \gamma^t \\ \eta^t & Z^t - I \end{bmatrix} \rightarrow K_1(C^*(E)/J_{(H,S)})$$

are group isomorphisms. Finally we let χ_* denote the group isomorphisms given directly by Proposition 3.8 for $C^*(E)$.

The theorem then follows from the ensuing six claims.

Claim 5.1. $\iota_* \circ \chi_0' = \chi_0 \circ \tilde{I}$.

Proof. If $v \in H$, then we have that

$$\begin{aligned} \chi_0 \circ \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -\Gamma^t \\ 0 & 0 & -\gamma^t \\ 0 & 0 & I - Z^t \end{bmatrix} \left(\mathbf{e}_v + \text{im} \begin{bmatrix} A^t - I \\ \alpha^t \\ 0 \end{bmatrix} \right) &= \chi_0 \left(\mathbf{e}_v + \text{im} \begin{bmatrix} A^t - I & X^t \\ \alpha^t & \xi^t \\ 0 & B^t - I \\ 0 & \beta^t \\ 0 & \eta^t \end{bmatrix} \right) \\ &= [p_v]_0 = [\iota(\phi(p_v))]_0 \\ &= \iota_* \circ \chi_0' \left(\mathbf{e}_v + \text{im} \begin{bmatrix} A^t - I \\ \alpha^t \\ 0 \end{bmatrix} \right), \end{aligned}$$

and if $v_0 \in S$, the left hand side equals

$$\begin{aligned} &\chi_0 \left(\mathbf{e}_{v_0} - \sum_{\substack{s(e)=v_0 \\ r(e) \notin H}} \mathbf{e}_{r(e)} + \text{im} \begin{bmatrix} A^t - I & X^t \\ \alpha^t & \xi^t \\ 0 & B^t - I \\ 0 & \beta^t \\ 0 & \eta^t \end{bmatrix} \right) \\ &= [p_{v_0}]_0 - \sum_{\substack{s(e)=v_0 \\ r(e) \notin H}} [s_e s_e^*]_0 = [\iota(p_{v_0}^H)]_0 = [\iota(\phi(p_v))]_0 \\ &= \iota_* \circ \chi_0' \left(\mathbf{e}_v + \text{im} \begin{bmatrix} A^t - I \\ \alpha^t \\ 0 \end{bmatrix} \right). \end{aligned}$$

□

Claim 5.2. $\pi_* \circ \chi_0 = \chi_0'' \circ P_{345}$.

Proof. As above, we check the claim of each class given by \mathbf{e}_v . If $v \in H$, then both sides vanish. If $v \notin H$, both sides equal $[p_v]_0$. □

Claim 5.3. $\pi_* \circ \chi_1 = \chi_1'' \circ I_1 \circ P_2$.

Proof. Fix

$$\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} \in \ker \begin{bmatrix} A^t - I & X^t \\ \alpha^t & \xi^t \\ 0 & B^t - I \\ 0 & \beta^t \\ 0 & \eta^t \end{bmatrix}.$$

Then $\begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix} \in \ker \begin{bmatrix} B^t - I & \Gamma^t \\ \beta^t & \gamma^t \\ \eta^t & Z^t - I \end{bmatrix}$ and we furthermore have that $L_{\begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix}}^+ \subseteq L_{\mathbf{x}}^+$ and $L_{\begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix}}^- \subseteq L_{\mathbf{x}}^-$. Thus if we let h be the number of elements in $L_{\mathbf{x}}^+$ (and in $L_{\mathbf{x}}^-$), and we let h' denote the number of elements in $L_{\begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix}}^+$ (and in $L_{\begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix}}^-$), then we can choose the bijections

$$[\cdot] : L_{\mathbf{x}}^+ \rightarrow \{1, \dots, h\} \quad \langle \cdot \rangle : L_{\mathbf{x}}^- \rightarrow \{1, \dots, h\}$$

and

$$[\cdot]' : L_{\begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix}}^+ \rightarrow \{1, \dots, h'\} \quad \langle \cdot \rangle' : L_{\begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix}}^- \rightarrow \{1, \dots, h'\}$$

such that $[\cdot]$ is an extension of $[\cdot]'$, and $\langle \cdot \rangle$ is an extension of $\langle \cdot \rangle'$. We then have that

$$\begin{aligned} \pi(\mathbf{V}_{\mathbf{x}}) &= \pi \left(\sum_{\substack{1 \leq i \leq x_w \\ s(e)=w}} s_e \mathbf{E}_{[w,i], \langle e, i \rangle} + \sum_{\substack{1 \leq i \leq -x_w \\ s(e)=w}} s_e^* \mathbf{E}_{[e,i], \langle w, i \rangle} \right) \\ &= \sum_{\substack{1 \leq i \leq z_w \\ s(e)=w}} \pi(s_e) \mathbf{E}_{[w,i], \langle e, i \rangle} + \sum_{\substack{1 \leq i \leq z_w \\ s(e)=w}} \pi(s_e^*) \mathbf{E}_{[e,i], \langle w, i \rangle} \\ &= \psi \left(\sum_{\substack{1 \leq i \leq z_w \\ s(e)=w}} s_e \mathbf{E}_{[w,i], \langle e, i \rangle} + \sum_{\substack{1 \leq i \leq -z_w \\ s(e)=w}} s_e^* \mathbf{E}_{[e,i], \langle w, i \rangle} \right) \\ &= \psi \left(\mathbf{V}_{\begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix}} \right) \end{aligned}$$

since $s_e \in J_{(H,S)} = \ker \pi$ when $s(e)$ (and thus $r(e)$) lies in H , and $z_w = x_w$ when $w \notin H$. A similar computation for $\mathbf{P}_{\mathbf{x}}$ shows that $\pi(\mathbf{P}_{\mathbf{x}}) = \psi \left(\mathbf{P}_{\begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix}} \right)$.

Thus $\pi(\mathbf{U}_{\mathbf{x}}) = \psi \left(\mathbf{U}_{\begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix}} \right)$ and

$$\pi_* \circ \chi_1(\mathbf{x}) = [\pi(\mathbf{U}_{\mathbf{x}})]_1 = [\psi(\mathbf{U}_{\begin{bmatrix} \mathbf{z} \\ 0 \end{bmatrix}})]_1 = \chi''([\mathbf{z}]) = \chi_1'' \circ I_1 \circ P_2(\mathbf{x}).$$

□

Claim 5.4. $\iota_* \circ \chi_1' = \chi_1 \circ I_1$.

Proof. Fix $\mathbf{x} \in \ker \begin{bmatrix} A^t - I \\ \alpha^t \\ 0 \end{bmatrix}$. This follows like in Claim 5.3 by choosing the bijections

$$[\cdot] : L_{\begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}}^+ \rightarrow \{1, \dots, h\} \quad [\cdot] : L_{\mathbf{x}}^+ \rightarrow \{1, \dots, h\}$$

and

$$\langle \cdot \rangle : L_{\begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}}^- \rightarrow \{1, \dots, h\} \quad \langle \cdot \rangle : L_{\mathbf{x}}^- \rightarrow \{1, \dots, h\}$$

to be pairwise equal. \square

Claim 5.5. $\partial_0 = 0$.

Proof. It follows from Claim 5.4 that $\iota_* : K_1(J_{(H,S)}) \rightarrow K_1(C^*(E))$ is injective. Thus $\text{im}(\partial_0) = 0$ from which it follows that $\partial_0 = 0$. \square

Claim 5.6. $\partial_1 \circ \chi_1'' = \chi_0' \circ \begin{bmatrix} X^t & 0 \\ \xi^t & 0 \\ 0 & I \end{bmatrix}$.

Proof. Fix $\mathbf{x} = \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} \in \ker \begin{bmatrix} B^t - 1 & \Gamma^t \\ \beta^t & \gamma^t \\ \eta^t & Z^t - I \end{bmatrix}$. We lift $\psi(\mathbf{V}_{\mathbf{x}})$ and $\psi(\mathbf{P}_{\mathbf{x}})$ to

$$\widehat{\mathbf{V}}_{\mathbf{x}} = \sum_{\substack{1 \leq i \leq x_w \\ s(e)=w, r(e) \notin H}} s_e \mathbf{E}_{[w,i], \langle e, i \rangle} + \sum_{\substack{1 \leq i \leq -x_w \\ s(e)=w, r(e) \notin H}} s_e^* \mathbf{E}_{[e,i], \langle w, i \rangle}$$

and

$$\widehat{\mathbf{P}}_{\mathbf{x}} = \sum_{1 \leq i \leq x_w} p_w \mathbf{E}_{[w,i], [w,i]} + \sum_{\substack{1 \leq i \leq -x_w \\ s(e)=w, r(e)=v \\ v \notin H}} p_v \mathbf{E}_{[e,i], [e,i]},$$

respectively, in $\mathbf{M}_h(C^*(E))$. Since $\{s_e, p_v : e \in r^{-1}(E^0 \setminus H), v \in E^0 \setminus H\}$ is a Toeplitz-Cuntz-Krieger $(E \setminus H)$ -family, it follows from Lemma 3.5 that $\widehat{\mathbf{V}}_{\mathbf{x}}$ is a partial isometry and that $\widehat{\mathbf{P}}_{\mathbf{x}} \widehat{\mathbf{V}}_{\mathbf{x}} = \widehat{\mathbf{V}}_{\mathbf{x}} \widehat{\mathbf{P}}_{\mathbf{x}} = \widehat{\mathbf{V}}_{\mathbf{x}}$. It follows that also $\widehat{\mathbf{U}}_{\mathbf{x}} := \widehat{\mathbf{V}}_{\mathbf{x}} + (1 - \widehat{\mathbf{P}}_{\mathbf{x}})$ is a partial isometry. Hence, to compute the value of the index map on $[\mathbf{U}_{\mathbf{x}}]_1$, we just need to compute the defect of $\widehat{\mathbf{U}}_{\mathbf{x}}$ in $K_0(J_{(H,S)})$, cf. [14, Proposition 9.2.2]. We have, using Lemma 3.4, that

$$\begin{aligned} 1 - \widehat{\mathbf{U}}_{\mathbf{x}} \widehat{\mathbf{U}}_{\mathbf{x}}^* &= \widehat{\mathbf{P}}_{\mathbf{x}} - \widehat{\mathbf{V}}_{\mathbf{x}} \widehat{\mathbf{V}}_{\mathbf{x}}^* \\ &= \sum_{1 \leq i \leq x_w} \left(p_w - \sum_{\substack{s(e)=w, r(e)=v \\ v \notin H}} s_e s_e^* \right) \mathbf{E}_{[w,i], [w,i]} \\ &= \sum_{1 \leq i \leq y_w} \left(\sum_{s(e)=w, r(e)=v} s_e s_e^* - \sum_{\substack{s(e)=w, r(e)=v \\ v \notin H}} s_e s_e^* \right) \mathbf{E}_{[w,i], [w,i]} \end{aligned}$$

$$\begin{aligned}
& + \sum_{1 \leq i \leq z_{v_0}} \left(p_{v_0} - \sum_{s(e)=v_0, r(e) \notin H} s_e s_e^* \right) \mathbb{E}_{[v_0, i], [v_0, i]} \\
& = \sum_{1 \leq i \leq y_w} \sum_{\substack{s(e)=w, r(e)=v \\ v \in H}} s_e s_e^* \mathbb{E}_{[w, i], [w, i]} + \sum_{1 \leq i \leq z_{v_0}} p_{v_0}^H \mathbb{E}_{[v_0, i], [v_0, i]}.
\end{aligned}$$

Passing to the K_0 -group and using that $s_e \in C^*(E_H)$, we get

$$\begin{aligned}
[1 - \widehat{U}_{\mathbf{x}} \widehat{U}_{\mathbf{x}}^*]_0 &= \sum_{1 \leq i \leq y_w} \sum_{\substack{s(e)=w, r(e)=v \\ v \in H}} [s_e s_e^*]_0 + \sum_{1 \leq i \leq z_{v_0}} [p_{v_0}^H]_0 \\
&= \sum_{0 < y_w} y_w \sum_{\substack{s(e)=w, r(e)=v \\ v \in H}} [s_e^* s_e]_0 + \sum_{0 < z_{v_0}} z_{v_0} [p_{v_0}^H]_0 \\
&= \sum_{0 < y_w} y_w \sum_{\substack{s(e)=w, r(e)=v \\ v \in H}} [p_v]_0 + \sum_{0 < z_{v_0}} z_{v_0} [p_{v_0}^H]_0.
\end{aligned}$$

The computation for $1 - \widehat{U}_{\mathbf{x}}^* \widehat{U}_{\mathbf{x}}$ is similar, and we get

$$[1 - \widehat{U}_{\mathbf{x}} \widehat{U}_{\mathbf{x}}^*]_0 - [1 - \widehat{U}_{\mathbf{x}}^* \widehat{U}_{\mathbf{x}}]_0 = \sum_{0 \neq y_w} y_w \sum_{\substack{s(e)=w, r(e)=v \\ v \in H}} [p_v]_0 + \sum_{z_{v_0} \neq 0} z_{v_0} [p_{v_0}^H]_0.$$

By comparison,

$$\begin{aligned}
\chi'_0 \left(\begin{bmatrix} X^t & 0 \\ \xi^t & 0 \\ 0 & I \end{bmatrix} \mathbf{x} \right) &= \chi'_0 \left(\begin{bmatrix} X^t \mathbf{y} \\ \xi^t \mathbf{y} \\ \mathbf{z} \end{bmatrix} + \text{im} \begin{bmatrix} A^t - 1 \\ \alpha^t \\ 0 \end{bmatrix} \right) \\
&= \sum_{y_w \neq 0} y_w \left(\sum_{v \in E_{\text{reg}}^0 \cap H} X_{v,w} [p_v]_0 + \sum_{v \in E_{\text{sing}}^0 \cap H} \xi_{v,w} [p_v]_0 \right) \\
&\quad + \sum_{z_{v_0} \neq 0} z_{v_0} [p_{v_0}^H]_0 \\
&= \sum_{x_w \neq 0} y_w \left(\sum_{\substack{s(e)=w, r(e)=v \\ v \in E_{\text{reg}}^0 \cap H}} [p_v]_0 + \sum_{\substack{s(e)=w, r(e)=v \\ v \in E_{\text{sing}}^0 \cap H}} [p_v]_0 \right) \\
&\quad + \sum_{z_{v_0} \neq 0} z_{v_0} [p_{v_0}^H]_0 \\
&= \sum_{x_w \neq 0} y_w \sum_{\substack{s(e)=w, r(e)=v \\ v \in H}} [p_v]_0 + \sum_{z_{v_0} \neq 0} z_{v_0} [p_{v_0}^H]_0,
\end{aligned}$$

completing the proof. \square

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DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE
AND TECHNOLOGY, NO-7491 TRONDHEIM, NORWAY

E-mail address: tokemeie@math.ntnu.no

DEPARTMENT FOR MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNI-
VERSITETSPARKEN 5, DK-2100 COPENHAGEN Ø, DENMARK

E-mail address: eilers@math.ku.dk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX 77204-
3008, USA

E-mail address: tomforde@math.uh.edu